Formalizing the Brouwer Fixed Point Theorem in Lean

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The Theorem

100 Theorems Benchmark

List of theorems tracking which have been formalized in which language.

“Benchmark” for the maturity of a mathematical formalization community, maintained by Freek Wiedijk.

Brouwer Fixed Point Theorem

(Brouwer, 1911) Let K be a nonempty, compact, convex subset of Euclidean space. Then any continuous mapping $f : K \rightarrow K$ admits a fixed point, i.e. there is some $a \in K$ such that $f(a) = a$.

Proof (informal):

1. A nonempty compact, convex set is homeomorphic to a closed ball.
2. From a fixpoint-free $B^n \to B^n$ we cook up a retraction $r : B^n \to S^n$.
3. Categorically, r is a split epi.
4. Split epimorphisms are preserved by functors!
5. Then $r_* : \tilde{H}_*(B^n) \to \tilde{H}_*(S^n)$ is a split epi. In particular it's surjective.
6. But $\tilde{H}_*(B^n) \cong 0$ while $\tilde{H}_*(S^n) \cong \mathbb{Z}$, so we obtain a contradiction!

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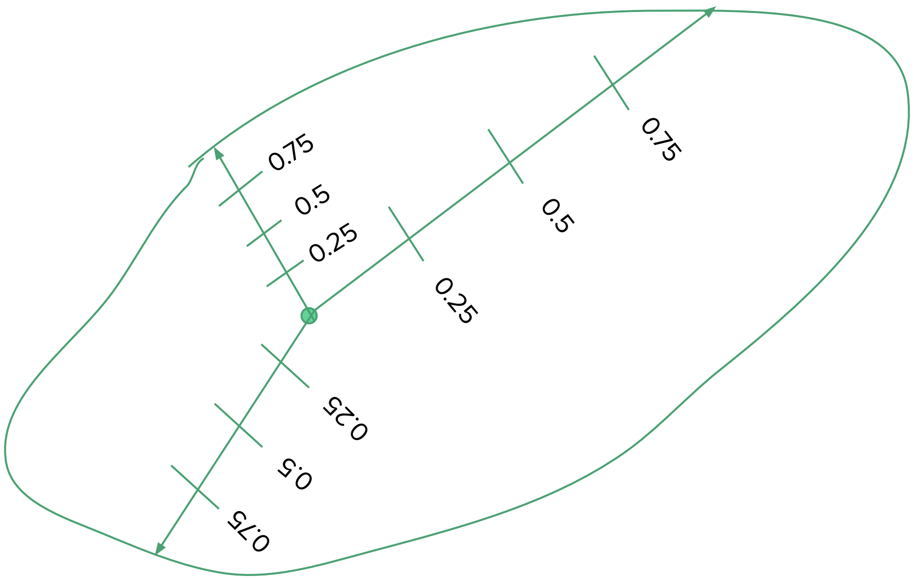
Convex bodies

A nonempty compact, convex set is homeomorphic to a closed ball.



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The retraction

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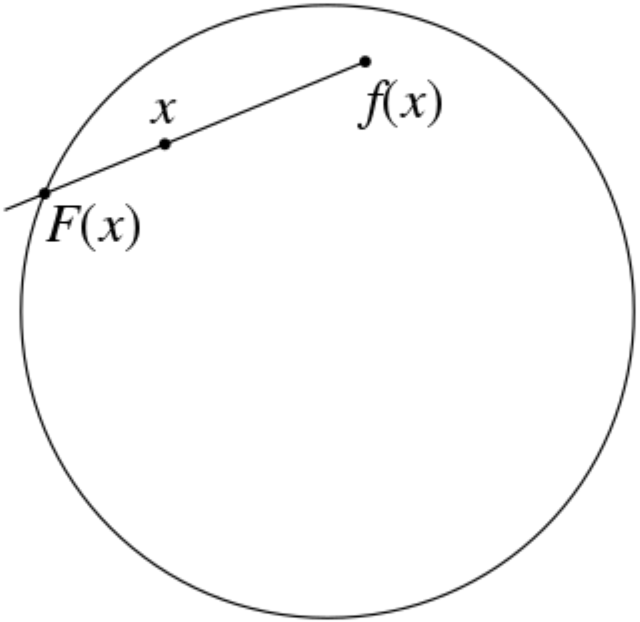

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Singular Homology

The definition

```
noncomputable
def free_complex_on_sset (R : Type*) [comm_ring R] : sSet ⥤ chain_complex (Module R) ℕ :=
  ((simplicial_object.whiskering _ _).obj (Module.free R)) ⋙ alternating_face_map_complex _

noncomputable
def singular_chain_complex (R : Type*) [comm_ring R] : Top ⥤ chain_complex (Module R) ℕ :=
  Top.to_sSet' ⋙ free_complex_on_sset R

noncomputable
def singular_chain_complex_of_pair (R : Type*) [comm_ring R]
  : arrow Top ⥤ chain_complex (Module R) ℕ :=
  category_theory.functor.map_arrow (singular_chain_complex R)
  ⋙ coker_functor (chain_complex (Module R) ℕ)

noncomputable
def singular_homology (R : Type*) [comm_ring R] (n : ℕ) : Top ⥤ Module R :=
  singular_chain_complex R ⋙ homology_functor _ _ n

noncomputable
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```

Design decisions

- Use custom `Top.to_sSet'` with “different” standard simplices
- Use a fixed commutative coefficient ring, not abelian group coefficients
- Define relative singular homology with respect to any map

Top.to_sSet'

```
def to_Top'_obj (x : simplex_category) := std_simplex ℝ x

def topological_simplex_alt_desc (n : simplex_category)
  : {f : n → nnreal | ∑ (i : n), f i = 1} ≃ₜ std_simplex ℝ n := {

def Top.to_sSet' : Top ⥤ sSet :=
colimit_adj.restricted_yoneda simplex_category.to_Top'

def Top.to_sSet_iso_to_sSet' : Top.to_sSet ≅ Top.to_sSet' :=
```

Coefficients in a (commutative) ring

- Allows singular homology to be a functor into R-Mod.
- Commutativity is just needed for Module.image (this should be fixed!)
- But even if ^ is fixed, probably still a bad design decision?!

Eilenberg-Steenrod Axioms

1. If $f, g : (X, A) \to (Y, B)$ are homotopic then the induced maps $f_*, g_* : H_i(X, A) \to H_i(Y, B)$ are equal.
2. Given an open cover $X = A \cup B$, the map $(A, A \cap B) \subseteq (X, B)$ induces an iso in homology.
3. If $X = \coprod_\alpha X_\alpha$, the comparison map $\bigoplus_\alpha H_i(X_\alpha) \to H_i(X)$ is an iso.
4. For any pair (X, A), the sequence

   $$\ldots \to H_{i+1}(X, A) \to H_i(A) \to H_i(X) \to H_i(X, A) \to \ldots$$

   is exact.

5. $H_i(pt) = 0$ for all $i > 0$.

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Homotopy invariance

Possible approaches:

- Explicitly define “prism operator” (Hatcher)
- Simplicial homotopies & $\mathrm{Sing}_\bullet$, Moore complex being monoidal
- Acyclic models theorem

Acyclic Models Theorem

Given a functor $F : C \to R\text{-Mod}$, a “basis” for F is a family of “models” $\{X_\lambda\}_{\lambda \in \Lambda}$ and elements $b_\lambda \in F(X_\lambda)$ such that for any $Y \in Obj(C)$, the family $\{F(f)(b_\lambda)\}_{\lambda \in \Lambda,\ f \in C(X\lambda,\ Y)}$ is a basis for the R-module $F(Y)$.

The case we care about: $C = Top$, $F = R\hat{}(\oplus\ Sing_i(-))$, $\{\Delta^n\}_{n \in \mathbb{N}}$, and b_n the identity map of Δ^n.

The point: A natural transformation $\eta : F \to G$ is specified by the values $a_\lambda = \eta_{X\lambda}(b_\lambda)$, with $\eta_Y(F(f)(b_\lambda)) = G(f)(\eta_{X\lambda}(b_\lambda))$.

Acyclic Models Theorem

Let $F_\bullet : C \to Ch^+(R\text{-}Mod)$ be a functor where each F_n is equipped with a basis. Another functor $G_\bullet : C \to Ch^+(R\text{-}Mod)$ is called *acyclic* if for all $n > 0$ and any model X for F_n we have $H_n(G_\bullet(X)) = 0$.

With this, any natural transformation $H_0(F_\bullet(-)) \to H_0(G_\bullet(-))$ lifts to a natural transformation $F_\bullet \to G_\bullet$, unique up to chain homotopy.

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AND $H_n(G_\bullet(X)) = 0$ for any model X for F_{n+1}.

With this, any natural transformation $H_0(F_\bullet(-)) \to H_0(G_\bullet(-))$ lifts to a natural transformation $F_\bullet \to G_\bullet$, unique up to chain homotopy.

The extra condition was **missing** in Dieck's book!

Acyclic Models Theorem

Let $F_\bullet$ be the singular chain complex functor and $G_\bullet(X) = F_\bullet(X \times I)$.

The inclusions of X into $X \times I$ as the height 0 and 1 cross sections give natural transformations $X \to X \times I$, which clearly induce the same map on 0th homology. For homotopy invariance we only need these maps!

Then by acyclic models we just need to show $G_\bullet$ is acyclic wrt $F_\bullet$, or that the homology of the contractible spaces $\Delta^n \times I$ vanishes in degree > 0.

Acyclic Models Theorem

Also, this method gives you the Eilenberg–Zilber theorem!

Excision

Proof: Easy homological algebra + barycentric subdivision (very very annoying).

```
lemma sufficient_barycentric_lands_in_cover (R : Type) [comm_ring R] {X : Top}
  (cov : set (set X)) (cov_is_open : ∀ s, s ∈ cov → is_open s) (hcov : ⋃₀ cov = ⊤) (n : ℕ)
  (C : ((singular_chain_complex R).obj X).X n)
  : ∃ k : ℕ, ((barycentric_subdivision_in_deg R n).app X) ^[k] C ∈ bounded_by_submodule R cov n :=
```

```
lemma subcomplex_inclusion_quasi_iso_of_pseudo_projection
  {C : homological_complex (Module.{v'} R) c}
  (M : Π (i : ι), submodule R (C.X i))
  (hcompat : ∀ i j, submodule.map (C.d i j) (M i) ≤ M j)
  (p : C ⟶ C) (s : homotopy (𝟙 C) p)
  (hp_eventual : ∀ i x, ∃ k, (p.f i)^[k] x ∈ M i)
  (hp : ∀ i, submodule.map (p.f i) (M i) ≤ M i)
  (hs : ∀ i j, submodule.map (s.hom i j) (M i) ≤ M j)
  : quasi_iso (Module.subcomplex_of_compatible_submodules_inclusion C M hcompat) :=
```

Excision

```
noncomputable
def barycentric_subdivision_in_deg (R : Type*) [comm_ring R]
  : Π (n : ℕ), (singular_chain_complex R ⋙ homological_complex.eval _ _ n)
               ⟶ (singular_chain_complex R ⋙ homological_complex.eval _ _ n)
| 0       := 𝟙 _
| (n + 1) := (singular_chain_complex_basis R (n + 1)).map_out
               (singular_chain_complex R ⋙ homological_complex.eval _ _ (n + 1))
               (λ _, @cone_construction_hom R _ (Top.of (topological_simplex (n + 1)))
                      (barycenter (n + 1))
                      ((convex_std_simplex ℝ (fin (n + 2))).contraction (barycenter (n + 1)))
                      n
                      ((barycentric_subdivision_in_deg n).app (Top.of (topological_simplex (n + 1)))
                        (((singular_chain_complex R).obj (Top.of (topological_simplex (n + 1)))).d
                          (n + 1) n
                          (simplex_to_chain (𝟙 (Top.of (topological_simplex (n + 1)))) R))))
```

Excision

```
lemma metric.lebesgue_number_lemma {M : Type*} [pseudo_metric_space M] (hCompact : compact_space M)
  (cov : set (set M)) (cov_open : ∀ s, s ∈ cov → is_open s) (hcov : ⋃₀ cov = ⊤)
  (cov_nonempty : cov.nonempty) -- if M is empty this can happen!
  : ∃ δ : nnreal, 0 < δ ∧ (∀ S : set M, metric.diam S < δ → ∃ U, U ∈ cov ∧ S ⊆ U) :=

lemma iterated_barycentric_subdivison_of_affine_simplex_bound_diam (R : Type) [comm_ring R]
  {ι : Type} [fintype ι] {D : set (ι → ℝ)} (hConvex : convex ℝ D)
  {n : ℕ} (vertices : fin (n + 1) → D) (k : ℕ)
  : ((barycentric_subdivision_in_deg R n).app (Top.of D))^[k]
      (simplex_to_chain (singular_simplex_of_vertices hConvex vertices) R)
  ∈ bounded_diam_submodule R D (((n : nnreal)/(n + 1 : nnreal))^k
                                    * ⟨@metric.diam D _ (set.range vertices), metric.diam_nonneg⟩) n
  ⊓ affine_submodule hConvex R n :=
```

Conclusion

By an ad-hoc inductive argument we can calculate $H_k(S^n)$!

Remaining work

- PR into mathlib & clean up codebase
- Singular cohomology, with cup product
- Show $H_n(S^n)$ is free on $[\Delta^n]$
- $H_i(X/A) = H_i(X, A)$ in nice cases
- Mayer-Vietoris
- Kunneth formula
- Simplicial/cellular homology
- Hurewicz theorem, $\pi_n(S^n) = \mathbb{Z}$
- Invariance of domain/dimension
- Lots and lots of work! Flip to a random page in Hatcher chapter 2, 3.

Questions